Optimal Physically Nonlinear Structures Made of Several Materials

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1. Abstract

Theoretical consideration of optimization problems for physically nonlinear hyperelastic structures is carried out. The structures are subjected to a single static "dead" loading, multi-material design approach is analyzed. Structural materials are supposed to be isotropic, with stress-strain relations being weakly concave. The problems considered are: mass minimization with prescribed structural stiffness, stiffness maximization with prescribed structural mass, mass minimization with constrained stresses. Optimality conditions for the problems are analyzed. Generalizations of Maxwell's and Michell's theorems for the considered structures are proved. Some regularities inherent in the third problem are analyzed using analytical example of 3-rod physically nonlinear truss made of 2 materials.

An algorithm for compliance decreasing in case of prescribed structural mass is proposed. Monotonicity property of the algorithm is proved. Numerical example is presented, corresponding results are decomposed on a basis of developed theoretical approaches.

2. Keywords

Optimization, Physical nonlinearity, Multi-material design, Algorithms.

3. Theoretical consideration

Theoretical consideration of optimization problems for physically nonlinear hyperelastic (PNH) structures is carried out. Strains and displacements are supposed to be small. The structures are subjected to a single static "dead" loading. Several structural materials may be simultaneously used for designing, it is assumed that prior to optimizing it is indicated which elements should be made of which material, actually the design purpose is optimal sizing. Structural materials are supposed to be isotropic, with stress-strain relations being weakly concave. Technology constraints on minimal allowable values of structural parameters are taken into account. The structural potential energy and the total complementary energy are considered as measures of structural stiffness and structural compliance, respectively. Buckling effects are neglected. Stress-strain fields are supposed to be uniform "along" variations of design parameters of structural members.

The first problem considered is the structural mass minimization problem with prescribed structural compliance and technology constraints. The second one is the structural compliance minimization problem with prescribed structural mass and technology constraints. Optimality conditions for the problems [1] are analyzed. According to the conditions the averaged element strain potential energy $\langle \Pi_i \rangle$ divided by the element density ρ_i should be constant over elements with passive technology constraints:

$$\langle \Pi_i \rangle / \rho_i = const \tag{1}$$

In case of active element technology constraint the above ratio should not exceed the constant.

The third problem considered is the problem of mass minimization with constrained absolute stress values and technology constraints. Maximal allowable stress value depends on element material.

Generalized multi-material Maxwell's theorem is presented. The **theorem** is written as follows: *for all PNH frameworks under a given system of applied forces, $\vec{P}_i$, acting at points with position vectors $\vec{r}_i$ (i=1,...,s) the following relation is valid:*

$$\sum_t P_t l_t - \sum_c P_c l_c = \sum_{i=1}^{s} \vec{P}_i \vec{r}_i = const \tag{2}$$

where P_t is the force in a tension member of length l_t and cross-section area F_t, ($-P_c$) is the force in a compressed member of length l_c and cross-section area F_c, $\sum_t$, $\sum_c$ are sums over the tension and compression members, respectively. For proving the theorem let's suppose that the following virtual displacement field is kinematically admissible:

$$\begin{aligned} \delta u &= e \cdot x \\ \delta v &= 0 \\ \delta w &= 0 \end{aligned} \tag{3}$$

where e is a small positive number, $\delta u, \delta v, \delta w$ are the displacements along x, y, z axes, respectively. It is the case when structural points located into the x=0 plane (or at least a point located in the origin) are fixed. The field is imposed on real one. Using the kinematic variational principle for PNH structures [2] written in the form of first variation

$$\sum_{\substack{all \\ rods}} \sigma_k \delta\varepsilon_k = \sum_{l=1}^{s} \vec{P}_l \vec{r}_l \tag{4}$$

we obtain the relation

$$\sum_t P_t l_t \cos^2 \varphi_x - \sum_c P_c l_c \cos^2 \varphi_x = \sum_{i=1}^{s} (P_x)_i (r_x)_i \tag{5}$$

where φ_x is the angle between the rod and the x axis, P_x, r_x are x components of $\vec{P}, \vec{r}$, respectively. Supposing that the structure is not supported and combining (summing) relations like (5) for y and z axes we finally obtain (2). Note that relations like (5) may be of help for analysis of supported structures.

In case of structures made of two structural materials (all tension members are made of one material and all compression members are made of another material) and fully-stressed design (FSD) conditions the following relation for the structural mass is obtained from (2):

$$M = \frac{1}{2}\left(\frac{\rho_t}{\sigma_t} + \frac{\rho_c}{\sigma_c}\right)\left(\sum_t P_t l_t + \sum_c P_c l_c\right) + \frac{1}{2}\left(\frac{\rho_t}{\sigma_t} - \frac{\rho_c}{\sigma_c}\right)\sum_{i=1}^{s} \vec{P}_i \vec{r}_i \tag{6}$$

where $\rho_t, \sigma_t, \rho_c, \sigma_c$ are the density and the maximal allowable stress level for tension and compression members, respectively.

The generalized Michell's theorem for the above 2-material structures is proved. The **theorem** is written as follows: *for all 2-material PNH frameworks S (all tension members are made of one material and all compression members are made of another material) which equilibrate a given force system within the domain D and have the stresses in all the tension and compression members* σ_t, σ_c, *respectively, the lightest S*, if exists, satisfies the following condition: there exists a virtual deformation of the domain D, with strains along the members of S* equal to ±e, where the sign agrees with that of the end load carried by the particular member, and such that no linear strain in D numerically exceeds e, which is a small positive number.* The proof of the theorem is based on the formula (6) and the kinematic variational principle [2]. The virtual deformation is considered as kinematically admissible displacement field variation δu^* for both S^* and arbitrary S. The potentials of external forces for the both structures corresponded to δu^* are equal, therefore the variations of the total strain energies are also equal. Due to the principle and properties of δu^* we obtain the following inequality:

$$\left(\sum_t P_t l_t + \sum_c P_c l_c\right)^* \le \sum_t P_t l_t + \sum_c P_c l_c \tag{7}$$

where asterisk corresponds to S^*. Observing (6) and (7) we obtain that the theorem is proved. It is obvious that S^* is also a minimal volume structure as compared to S, moreover the structure has the minimal total strain potential energy. The last statement follows from the expression for the energy $\tilde{\Pi}$ of FSD structure:

$$\tilde{\Pi} = \frac{1}{2}\left(\frac{\Pi_t}{\sigma_t} + \frac{\Pi_c}{\sigma_c}\right)\left(\sum_t P_t l_t + \sum_c P_c l_c\right) + \frac{1}{2}\left(\frac{\Pi_t}{\sigma_t} - \frac{\Pi_c}{\sigma_c}\right)\sum_{i=1}^{s} \vec{P}_i \vec{r}_i \tag{8}$$

where Π_t, Π_c are the specific strain energies for tension and compression members corresponded to σ_t, σ_c, respectively. Minimality of the total complementary energy of S^* is proved analogously. Therefore, the potential of external forces of S^* reaches its minimal value as compared to S. The potential as a sum of the two total energies is one more measure of structural compliance but the total complementary energy.

Actually the above theoretical results give rise to a recommendation to designers to choose materials for 2-material structures keeping in mind FSD conditions and the relation:

$$\Pi_t / \rho_t = \Pi_c / \rho_c \tag{9}$$

Some regularities inherent in the above third problem are analyzed on a basis of use of a model structure, namely, physically nonlinear 3-member plane truss made of two structural materials (see Fig.1). The structure is symmetric, rod 2 is made of one material, rods 1 having the same value of cross-section area are made of another one. It is obvious that due to the strain compatibility condition the rod strains $\varepsilon_i, i = 1,2$, satisfy the relation

$$\varepsilon_2 = 2\varepsilon_1 \tag{10}$$

where subscripts correspond to rod numbers. Analytical optimal solutions of the considered optimization problem are obtained. Theoretical analysis of the solutions is performed. It is supposed that nonlinear stress-strain laws for the above rods have initial linear-elastic parts with some values of the Young module. It is obtained that for certain combinations of structural parameters the optimal physically nonlinear structure is heavier than the optimal linear-elastic one (in the latter case the structure is supposed to have the above Young modules).

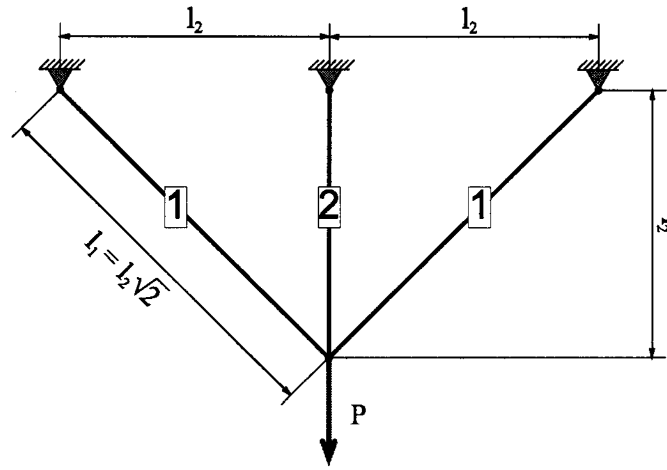

Fig. 1. Three-member truss

Let's restrict the consideration by the obvious case when material #2 is stronger than material #1 and the optimal linear-elastic structure has active technology constraint in rod 1 and active stress constraint in rod 2. Then the combinations are the following:

1) the optimal PNH structure has a reversed set of active constraints, namely the structure has active stress constraint in rod 1 and active technology constraint in rod 2;
2) the optimal PNH structure has active technology constraint in rod 1 and active stress constraint in rod 2, the stress value of rod 1 in linear-elastic case is greater than the stress value of the rod in the nonlinear case.

It is noted that in case of use of one material for producing the structure the optimal nonlinear one could not be heavier.

It is known that linear-elastic model is widely used for preliminary assessment of rational parameters of PNH structures, the former model usually results in some mass reserves of the structure to be designed. The performed analysis improves understanding of physical regularities when using the approach.

4. Algorithms

An algorithm for structural compliance decreasing is proposed. The algorithm is based on the above-mentioned optimality conditions (1) and the static variational principle for PNH structures [2]. Similar algorithms were proposed earlier for the case of structures made of one material [3].

One step of the algorithm consists of two sub-steps: forces within structural elements are supposed to be frozen and new values of structural parameters from minimization of the total complementary energy with prescribed structural mass are obtained, then strain compatibility conditions are satisfied. During the first sub-step the total complementary energy is obviously decreased. During the second sub-step the total complementary energy is decreased due to the static variational principle [2]. Hence we have proved that the algorithm provides monotonous decrease of structural compliance.

It is indicated that the algorithm may be also used for physically nonlinear structures described by models close to PNH one. Some variants of the algorithm are discussed.

It is also indicated that FSD approach may be accomplished by using the known stress-ratio type algorithm.

5. Numerical example

Physically nonlinear I-beam structure made of two structural materials is considered (see Fig. 2). In the Figure the loading of the structure is also demonstrated, namely the end section of the beam is loaded by the force distributed over the section. Numerical results for optimization of the structure are presented. The caps and the web of the beam are considered as produced from steel and aluminum, respectively. Maximal allowable stress level in caps is equal to 175 kg/sq.mm, the allowable stress level for web is equal to 50 kg/sq.mm. Stress-strain diagrams for both materials are approximated by bilinear curves.

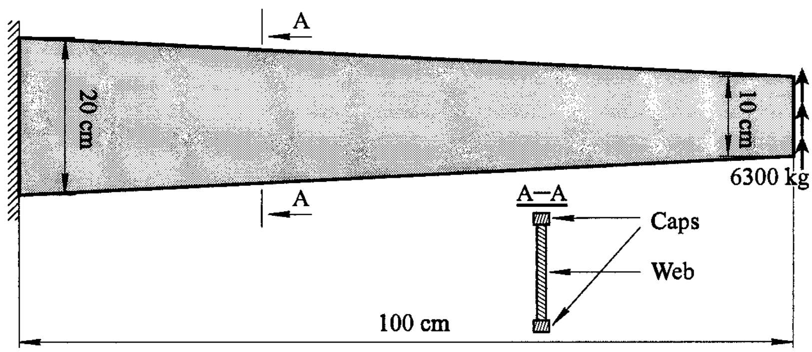

Fig. 2. Model I-beam

Analysis of the structure is made by the finite element method, the deformation plasticity theory being used for analysis of inelastic structural deforming beyond the yield limit. The FE model consists of rods and 2D-stress elements, the total number of the elements is equal to 168 (56 rods and 112 membranes). The stress-ratio type algorithm is used for the optimization, as well as the above-described compliance decreasing one. Fig. 3 demonstrates final designs (rod areas, membrane thickness values) obtained. Intensification of dithering intensity corresponds to increase of design parameter values.

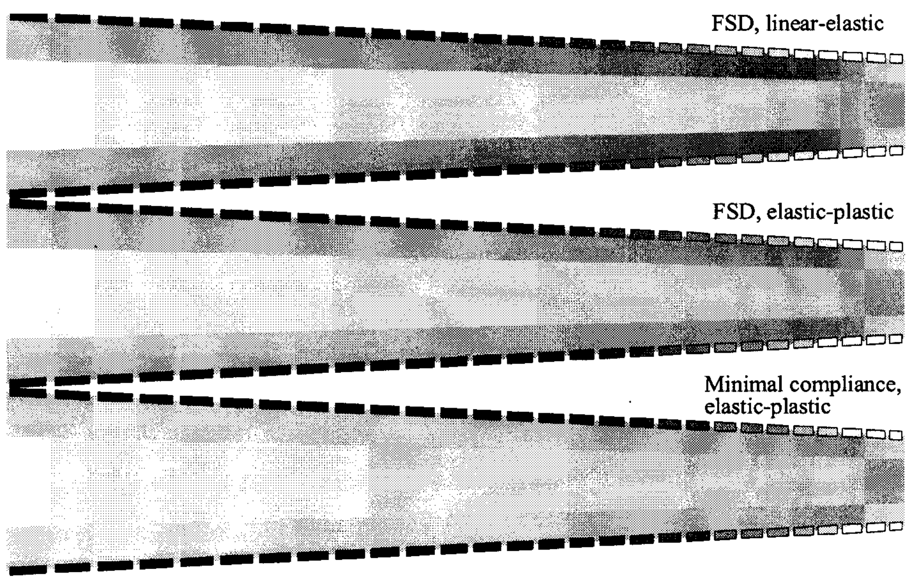

Fig. 3. Design variants.

Structural mass values obtained respectively are 2.2, 2.15, 2.15 kg for top to bottom. The bottom structure is close to an FSD one with stresses in caps being equal to 170.4 kg/sq.mm and averaged stresses in membranes being equal to 50.2 kg/sq.mm. Maximal displacement and the total complementary energy of the structure are monotonously decreasing during the optimization process. Final maximal structural displacement of the bottom structure is 14% less than the displacement of the elastic-plastic FSD one.

Observing the three structures we can see that there is an effect of some caps degeneration in the sections close to loaded one. At the same sections edge (top and bottom) membrane elements become stronger than ones close to the symmetry axis. Actually the elements create something like "additional caps". The effect is more significant in case of the linear-elastic FSD structure than in cases of two other ones, this is the reason of larger structural mass of the former structure.

The considered structures have a regular relationship between structural mass values of the linear-elastic and nonlinear structures optimized, namely the linear-elastic FSD structure is heavier than the physically nonlinear FSD one. The bottom design of Fig. 3 having the same mass value looks preferable due to less compliance value, greater material concentration in caps and less values of cap stresses.

For the case of 6% less value of the maximal allowable web stress level (47 kg/sq.mm) the obtained designs (rod areas, membrane thickness values) are presented in Fig. 4. Bottom design in the Figure corresponds to the case of 163.2 kg/sq.mm max allowable stress level in caps, in this case the criterion (1) is satisfied.

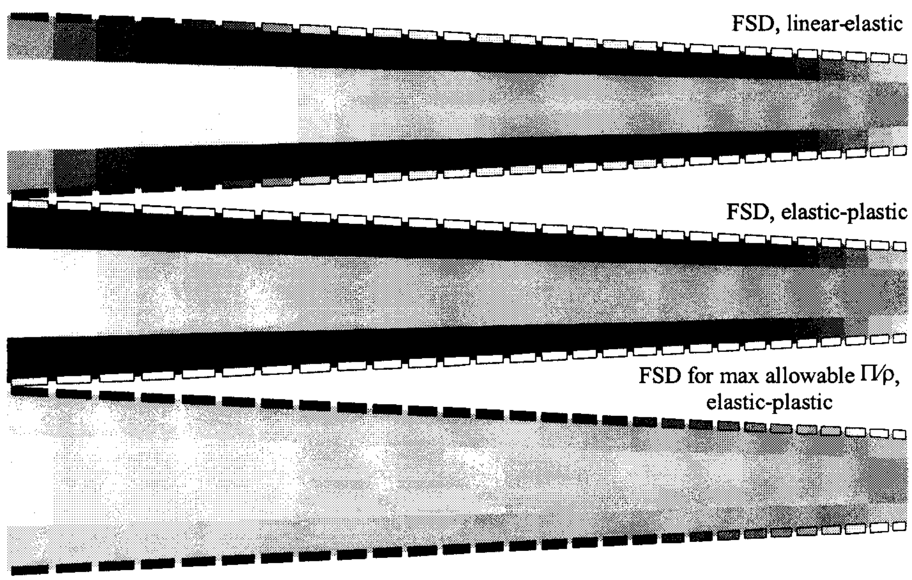

Fig. 4. Design variants, 6% less value of max allowable stress level.

Structural mass values for the design variants are 2.85, 3.29, 2.26 kg from top to bottom, respectively. Maximal displacement corresponding to the bottom variant is 22% less than the displacement corresponding to the middle one.

It follows from the Figure that the above-mentioned effect is stronger for the 2 upper design variants than for the structures of Fig. 3. We clearly see from the 2 upper design variants of Fig. 4 that caps of the whole beam degenerate with creating "additional caps" from the web material, the behavior being more expressed for elastic-plastic structure. In the irregular case of Fig. 4 the fully-stressed elastic-plastic structure is noticeably heavier than the linear-elastic one. The above-made analysis for 3-member truss (namely case 1 at the end of Section 3) gives us a physical explanation of the result. The bottom design variant also demonstrates a possibility to cure the irregular case by adjusting (decreasing!) maximal allowable stress level in steel caps. Observing design parameters of the variant we see that it corresponds to regular case of web degeneration. Moreover we obtain less value of maximal displacement for the variant.

6. References

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